Open Chemical Systems and Their Biological Function

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Dynamics and Thermodynamics of Stochastic Nonlinear (Mesoscopic) Systems

Mesoscopic description of physical and chemical systems:

- Gibbs (1870-1890s) – complex system in equilibrium in terms of ensembles
- Einstein, Smoluchowski, Langevin – simple motions, linear dynamics (1900-1910s)
- Kramers (1940) – emergent rare events in nonlinear systems
- Onsager (1953) – general linear dynamical theory

Mesoscopic description of physical and chemical systems:

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Mesoscopic description of a system - the mathematical tool:

- Kolmogorov (1933) – mathematical theory of random variables & stochastic processes

The Aim of Statistical Mechanics

``to develop a formalism from which one can deduce the macroscopic behavior of physical systems composed of a large number of molecules from a specification of the component molecular species, the laws of force which govern intermolecular interactions, and the nature of their surroundings.'' [Montroll and Green, *Ann. Rev. Phys. Chem*. (1954)]

The Aim of Statistical Physics (now half a century later)

To develop a formalism from which one can deduce the macroscopic behavior of complex systems from a specification of the components, the laws of force which govern dynamics, and the nature of their surroundings.

In Traditional Physics:

- A law of a force is an interaction between particles;

- However, as many physical chemists know well, *entropic force* is not a force between particles; in fact it is an emergent entity on a population level. The Fick's law makes no sense on an individual Brownian particle level;

- Still, a force is something that causes a system to change.

What is thermodynamics?

Thermodynamics deals with energy, entropy, their balance, and inter-relationships in complex systems
(no temperature in this talk).

What is Kolmogorov's Stochastic Process?

- It is a mathematical description of dynamics with "uncertainties". It has both a trajectory perspective and a population perspective. They are complementary; neither is a complete story.

- Classical dynamics of Newton and Laplace has singular distribution, quantum dynamics has distribution but no trajectory, stochastic process requires both.

For Stochastic Process with Continuous Paths

- Its trajectory can be described by a stochastic differential equation (generalized nonlinear Langevin equation)
- Its distribution is described by a Fokker-Planck (Kolmogorov forward) equation.

$$dx(t) = b(x)dt + \sigma(x)\,dB(t)$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\sigma^2(x)}{2}\frac{\partial f(x,t)}{\partial x}\right) - \frac{\partial}{\partial x}\left(b(x)f(x,t)\right)$$

For Stochastic Process with Discrete States & Jumps

- Its trajectory can be described by the Bortz-Kalos-Lebowitz-Gillespie algorithm
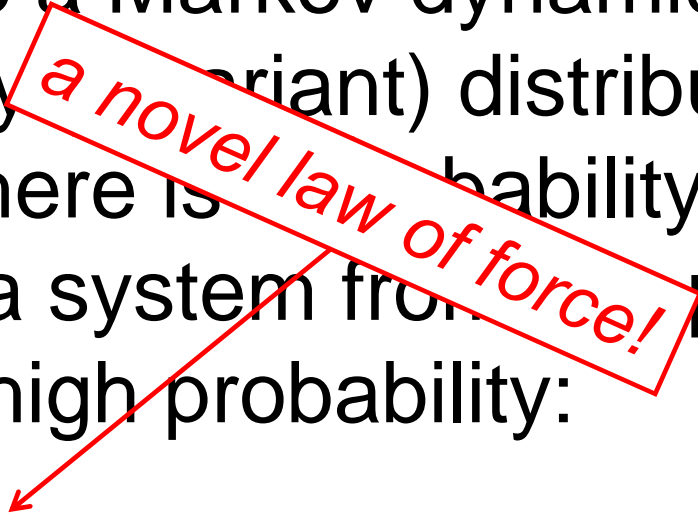- Its distribution is described by master equation

$$\Pr\left(\xi_{t+dt} = j \mid \xi_t = i\right) = q_{ij} dt \quad (i \neq j)$$

$$\frac{dp(n,t)}{dt} = \sum_m p(m,t) q_{mn} - p(n,t) q_{nm}$$

A disclaimer …

"Generalized" Energy, Entropy and Free Energy in a Markov System

Let us assume a Markov dynamics has a unique stationary (invariant) distribution. This means that there is a probability based "force" pushing a system from low probability to high probability:

a novel law of force!

$$E_n = -\ln p_n^{ss}$$

[Haken & Graham, Kubo *et al*., Nicolis & Lefevere, Ao]

"Generalized" Energy, Entropy and Free Energy – Cont.

Then one has the energetics of the system:

$$\overline{E}(t) = -\sum_n p_n(t) \ln p_n^{ss}, \quad S(t) = -\sum_n p_n(t) \ln p_n(t);$$

$$F(t) = \overline{E}(t) - S(t) = \sum_n p_n(t) \ln\left[\frac{p_n(t)}{p_n^{ss}}\right]$$

F(t) is also known relative entropy.

Then we uniquely have,

source

sink

$$\frac{dF(t)}{dt} = E_{in}(t) - e_p(t);$$

$$e_p(t) = \sum_{i,j} \left[p_i(t)q_{ij} - p_j(t)q_{ji} \right] \ln \left[\frac{p_i(t)q_{ij}}{p_j(t)q_{ji}} \right] \geq 0;$$

$$E_{in}(t) = \sum_{i,j} \left[p_i(t)q_{ij} - p_j(t)q_{ji} \right] \ln \left[\frac{p_i^{ss}q_{ij}}{p_j^{ss}q_{ji}} \right] \geq 0.$$

Non-negative energy input E_{in}

$$E_{in} = \frac{1}{2} \sum_{i,j} (p_i q_{ij} - p_j q_{ji}) \ln\left(\frac{p_i^s q_{ij}}{p_j^s q_{ji}}\right)$$

$$= \sum_{i,j} p_i q_{ij} \ln\left(\frac{p_i^s q_{ij}}{p_j^s q_{ji}}\right) = -\sum_{i,j} p_i q_{ij} \ln\left(\frac{p_j^s q_{ji}}{p_i^s q_{ij}}\right)$$

$$= -\sum_{i,j \neq i} p_i q_{ij} \ln\left(\frac{p_j^s q_{ji}}{p_i^s q_{ij}}\right) \geq \sum_{i,j \neq i} p_i q_{ij} \left(1 - \frac{p_j^s q_{ji}}{p_i^s q_{ij}}\right)$$

$$= \sum_{i,j} p_i q_{ij} \left(1 - \frac{p_j^s q_{ji}}{p_i^s q_{ij}}\right) = \sum_{i,j} \left(p_i q_{ij} - \frac{p_i p_j^s q_{ji}}{p_i^s}\right)$$

$$= \sum_i p_i \sum_j q_{ij} - \sum_i \frac{p_i}{p_i^s} \sum_j p_j^s q_{ji} = 0.$$

Energy balance equation for a subsystem is a generalization of energy conservation of an isolated system: We interpret this mathematical result as "the 1st Law of thermodynamics".

Furthermore, we have

$$\frac{dF(t)}{dt} \leq 0, \ or \ e_p \geq E_{in}$$

We interpret this mathematical result as "the 2nd Law of Thermodynamics".

Non-positive free energy change

$$\frac{dF}{dt} = \sum_j \frac{dp_j}{dt} \ln\left(\frac{p_j}{p_j^s}\right) = \sum_{i,j} (p_i q_{ij} - p_j q_{ji}) \ln\left(\frac{p_j}{p_j^s}\right)$$

$$= \sum_{i,j} \left[p_i q_{ij} \ln\left(\frac{p_j}{p_j^s}\right) - p_i q_{ij} \ln\left(\frac{p_i}{p_i^s}\right) \right]$$

$$= \sum_{i,j} p_i q_{ij} \ln\left(\frac{p_j p_i^s}{p_j^s p_i}\right) = \sum_{i,j \neq i} p_i q_{ij} \ln\left(\frac{p_j p_i^s}{p_j^s p_i}\right)$$

$$\leq \sum_{i,j \neq i} p_i q_{ij} \left(\frac{p_j p_i^s}{p_j^s p_i} - 1\right) = \sum_{i,j} p_i q_{ij} \left(\frac{p_j p_i^s}{p_j^s p_i} - 1\right)$$

$$= \sum_{i,j} \frac{p_j p_i^s q_{ij}}{p_j^s} - p_i q_{ij} = \sum_j \frac{p_j}{p_j^s} \sum_i p_i^s q_{ij} - \sum_i p_i \sum_j q_{ij}$$

$$= 0.$$

An alternative interpretation:

Boltzmann's thesis known as house-keeping hea_hk
(Oono and Paniconi, 1998)

$$e_p = -\frac{dF(t)}{dt} + Q_{hk} \geq 0.$$

Prigogine's thesis

Two origins of irreversibility, e_p is the total entropy production.

For System with Detailed Balance:

$$E_{in}(t) = \sum_{i,j} \left[p_i(t)q_{ij} - p_j(t)q_{ji} \right] \ln \left[\frac{p_i^{ss} q_{ij}}{p_j^{ss} q_{ji}} \right] = 0.$$

Then,

$$\frac{dF(t)}{dt} = -e_p(t);$$

This is known to Gibbs: While for canonical ensemble the appropriate potential function is free energy, not entropy, but the origin of 2nd Law is still entropy production.

For System with detailed balance and uniform stationary distribution:

$$\overline{E}(t) = -\sum_n p_n(t) \ln p_n^{ss} = const.$$

$$F(t) = -S(t) + const.$$

This is a microcanonical ensemble.

Entropy Balance Equation
(de Groot and Mazur, 1962)

$$\frac{dS(t)}{dt} = e_p(t) - h_d(t) = \frac{d_i S}{dt} + \frac{d_e S}{dt};$$

$$h_d(t) = \sum_{i,j} \left[p_i(t) q_{ij} - p_j(t) q_{ji} \right] \ln \left[\frac{q_{ij}}{q_{ji}} \right].$$

[Onsager (1931), Eckart & Bridgman (1940),
Prigogine (1945), de Groot (1951),
Bergmann & Lebowitz (1955)]

Physical origins of entropy production, free energy dissipation, and their mathematical representations

Hao Ge[1,*] and Hong Qian[2,1,†]

Now Nonlinear Stochastic Dynamics …

We consider Markov processes with continuous path:

$$dx(t) = b(x)dt + \sigma(x)\,dB(t)$$

$$\frac{\partial f(x,t)}{\partial t} = \nabla \cdot \left(A(x)\nabla f(x) - b(x)f(x) \right) = L[f]$$

We would like to introduce a symmetric-anti-symmetric decomposition for $L[f]$. To do that, we introduce an inner product:

$$(\psi, \varphi) = \int \psi(x)\, \varphi(x) \left(f^{ss}(x) \right)^{-1} dx$$

Then we have

$$L(\varphi) = \nabla \cdot \left(A\nabla\varphi \right) - \nabla \cdot \left(b(x)\varphi(x) \right),$$

$$= L_S(\varphi) + L_A(\varphi)$$

$$L_S(\varphi) = \nabla \cdot \left(A\nabla\varphi - \left(A\nabla \ln f^{ss}(x) \right) \varphi(x) \right),$$

$$L_A(\varphi) = \nabla \cdot \left(\left(A\nabla \ln f^{ss}(x) - b(x) \right) \varphi(x) \right),$$

$$\frac{dx}{dt} = j(x), \ \nabla \cdot \left(j(x) f^{ss}(x) \right) = 0.$$

More importantly

For dynamics with only the symmetric part

$$\frac{\partial u}{\partial t} = L_S[u], \ \ E_{in}(t) \equiv 0.$$

For dynamics with only the Anti-symmetric part

$$\frac{\partial u}{\partial t} = L_A[u], \ \ \frac{dF(t)}{dt} \equiv 0.$$

The symmetric and anti-symmetric parts of the dynamics generalize nicely Fourier's dissipative dynamics (heat equation) and Newton's conservative dynamics (volume preserving).

PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore *IS. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheseos
Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR·
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Jussu Societatis Regiæ ac Typis *Josephi Streater.* Prostat apud
plures Bibliopolas. *Anno* MDCLXXXVII.

THE

ANALYTICAL THEORY OF HEAT

BY

JOSEPH FOURIER.

TRANSLATED, WITH NOTES,

BY

ALEXANDER FREEMAN, M.A.,
FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

EDITED FOR THE SYNDICS OF THE UNIVERSITY PRESS.

I. Prigogine
Nobel Lecture (1977)

[L]et us emphasize that one hundred fifty years after its formulation, the second law of thermodynamics still appears more as a program than a well defined theory in the usual sense, as nothing precise (except the sign) is said about the entropy production. Even the range of validity of this inequality is left unspecified.

Our Major Claim

For complex systems, the thermodynamic laws are consequences of (nonlinear) dynamical descriptions of a system with stochastic. The mathematical theory of stochastic processes for mesoscopic systems supports a (equilibrium and nonequilibrium) thermodynamic structure which consists of both 1st and 2nd Laws.

The validity of non-equilibirum thermodynamics, therefore, no longer relies on "local equilibrium assumption" as in the past. Rather the burden is shifted to the validity of a Markovian description of a natural process, be it from physics, biology, economics, or sociology. There is absolutely no assumption on linearity!

Thermodynamic relations are not natural laws; they are mathematical theorems with applications in nature. Dynamics is more fundamental; it is the model for natural phenomena.

Thermodynamic relations are not natural laws; they are mathematical theorems with applications in nature. Dynamics is more fundamental; it is the model for natural phenomena. Thermodynamics, however, is absolute.

Two further developments in the making:

(1) Temperature as a measure of "distance" between a mesoscopic system and its deterministic limit; zeroth law and third laws;

(2) Stochastic Partial Differential Equations with real physical space (i.e., reaction-diffusion)

- Pope-Ching formalism;
- This is essentially the fluctuating hydrodynamic formalism;
- Again, Langevin and Fokker-Planck are just two different perspectives of a same dynamic process.

Stochastic Physics in Biology

Foundations and Current Trends

January 13-18, 2013
Four Points Sheraton / Holiday Inn Express
Ventura, CA

Chair:
Hong Qian

Vice Chair:
Steven J. Altschuler

Application Deadline

Applications for this meeting must be submitted by **December 16, 2012**. Please apply early, as some meetings become oversubscribed (full) before this deadline. If the meeting is oversubscribed, it will be stated here. *Note*: Applications for oversubscribed meetings will only be considered by the Conference Chair if more seats become available due to cancellations.

Thank You!