Self-propelling particles as a new challenge for the Nonequilibrium Statistical Mechanics

V. Kulinskii and O. Chepizhko
Odessa I.I.Mechnikov Nationala University,
Department of Theoretical Physics,
2 Dvoryanskaya Str., Odessa 65026, Ukraine
chepizhko@onu.edu.ua

1. Introduction

Statistical physics has proven to be a fruitful framework to describe phenomena outside the realm of traditional physics. Recent years have witnessed an attempt by physicists to study collective phenomena emerging from the interactions of individualsas elementary units in social structures.(Castellano C. at al., Statistical physics of social dynamics, Rev. Mod. Phys., 2009, **81**, p. 591).

The **Vicsek** model (VM) is a simple yet productive way to describe the behavior of the system of self-propelled particles which interact via exchange of information. It gave an intense impact for the research in this new field of physics. In this model each particle tends to move in the way its neighbors do. Ordered motion emerges in such system at high enough densities. Some stochastic perturbation (noise) may be added to the system and this can lead to order-disorder transition. We show that type of phase transition in the Vicsek model (VM) for self-propelled particles depends on the type of noise introduced to the system. Such models should represent nontrivial hydrodynamics because of nonholonomic character of the microscopic dynamics. We use the microscopic phase density functional (MPDF) approach to derive the equation for the distribution function. The numerical experiment on the modelling of Couette flow reveals no velocity profile which signifies the viscose-free hydrodynamics of such system. In contrast to the molecular systems the regular approach from the nonequilibrium statistical mechanics is an open problem for such kind of systems.

2. The basic models

The dynamical equation in continuous time limit for the Vicsek model has been considered earlier. It has simple form:

$$\frac{d}{dt}\mathbf{v}_i = \boldsymbol{\omega}_{\mathbf{v}_i} \times \mathbf{v}_i\,, \tag{1}$$

where $\boldsymbol{\omega}_{\mathbf{v}_i}$ is the “angular velocity“ of i-th particle. This angular velocity depends on the velocities of neighboring particles. The self-propelling force and the frictional force are assumed to balance each other.

It should be noted that the same situation is well known for another model of synchronization - the **Kuramoto** (KM) model determined by the following dynamical equations for the oscillator phases:

$$\dot{\theta}_i = \omega_i + k\sum_{\langle i,j\rangle} \sin\left(\,\theta_j - \theta_i\,\right) = \omega_i + k\,N_i\,u_i\,\sin\left(\,\psi_i - \theta_i\,\right), \tag{2}$$

where k - is the interaction strength, ψ_i is the average local phase of the nearest oscillators.

Yet in the KM the type of the transition depends on the distribution function $g(\omega)$ of the proper frequencies of the oscillators. We consider 2D case and use the following definition of the local order parameter:

$$r_i\,e^{i\,\psi_i} = \frac{1}{N_i}\sum_{\langle i,j\rangle} e^{i\,\theta_j}\,. \tag{3}$$

The same definition is used for the KM (see Eq. 2). Indeed, let the angle θ_i characterize the direction of the velocity of i-th particle, then $\boldsymbol{\omega}_{\mathbf{v}_i} = \dot{\theta}_i$ and the equation of motion takes the form, as it was shown:

$$\dot{\theta}_i = \dot{\psi}_i + A\,\sin\left(\,\psi_i - \theta_i\,\right)\,. \tag{4}$$

The main difference between (2) and (4) is the form of the first term. In the KM it is determined once and forever by the distribution function $g(\omega)$. In the VM thi term has collective contribution, so VM tends to synchronize more efficiently. *Thus we can state that these models are isomorphic at least in the mean field approximation.*

First let us consider the case of scalar noise which can be modeled by inclusion of the common Langevin source into the equation of motion with $g(\omega) = \delta(\omega)$:

$$d\theta_i = -k\,r\sin(\theta_i)\,dt + dw_i(t)\,. \tag{5}$$

Here $w_i(t)$ is the Wiener process stands for the random increment of the angle. In the mean-field approximation one easily comes to the self-consistent equation for the stationary value of the order parameter:

$$r = \frac{I_1(\frac{Kr}{D})}{I_0(\frac{Kr}{D})}\,. \tag{6}$$

In such case the nontrivial solution appears continuously from the trivial one. As the second case we consider the vectorial perturbation of the VM introduced by Gregoire and Chaté. This perturbation corresponds to the stochastic deviation of the direction of motion for i-th particle due to addition of the random vector $\boldsymbol{\xi}_i$ with $|\boldsymbol{\xi}_i| = \xi\,N_i$.

The self-consistent mean-field equation for the order parameter:

$$r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left(\frac{r}{\xi} + \cos\alpha\right)\,d\alpha}{\sqrt{1 + 2\frac{r}{\xi}\cos\alpha + \left(\frac{r}{\xi}\right)^2}} \tag{7}$$

remarkably coincides with that for the order parameter of the Kuramoto-like model augmented with the phase pinning studied by Strogatz et al. By the comparison of the results we can state that ξ is equivalent to the pinning strength. In such case of the vectorial noise we have subcritical bifurcation of the solution where there is the discontinuous jump between ordered and disordered motion because of the existence of the metastable state. One can expect that if some new parameter is added to the system, then tricritical behavior can appear. Note that the tricritical behavior is also demonstrated by the Kuramoto model. The simplest variant is the statistical mixing of the vector and the scalar noises. The corresponding self-consistent equation is of the form:

$$r = \eta\,F_v(r) + (1-\eta)\,F_s(r)\,. \tag{8}$$

Here the right hand side of (8) corresponds to the statistically mixed probability density for the angle of direction:

$$f_{mix}(\theta, r) = \eta f_v(\theta, r) + (1-\eta) f_s(\theta, r) \tag{9}$$

where f_s nd f_v are distribution functions for scalar and vector noises respectively and $0 \le \eta \le 1$ denotes the parameter of statistical mixing.

The analysis of Eq. (8) shows that the bifurcation of its solutions with varying parameter η corresponds to the tricritical behavior and there is an area where two stable regimes are possible. When additional parameter was added the possibility appeared for both behaviors, supercritical and subcritical, to exist at one time. Indeed for the case of “mixed“ noise there is a region of values K and η where two nontrivial stable regimes exist simultaneously. The complete phase diagram for the mixed noise is given by Fig. 1.

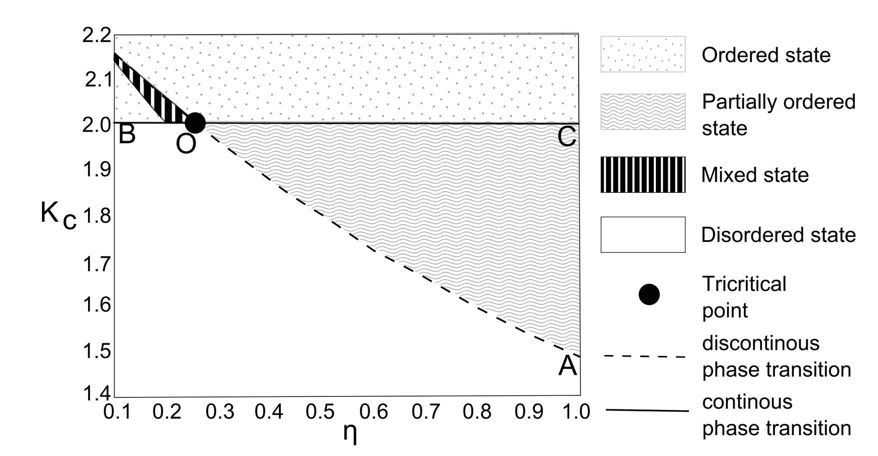

Figure 1: *Phase diagram of the system with mixed noise*

3. Kinetic equation

We start with the derivation of the equation of motion for the MPDF. Though such derivation is standard task we perform it here to stress the importance of the fact that the interaction between particles is not of potential type. For a system of N particles with the positions and the velocities $\mathbf{r}_i, \mathbf{v}_i$ correspondingly one can use the standard definition of the microscopic phase density:

$$\mathcal{N}(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i(t))\,\delta(\mathbf{v} - \mathbf{v}_i(t)) \tag{10}$$

with the standard normalization condition:

$$\int \mathcal{N}(x,t)\,dx = N\,. \tag{11}$$

We use the notation $x = (\mathbf{r}, \mathbf{v})$ for the point in the phase space. Since $\mathcal{N}$ is a functional we consider any proper function $f(x,t)$ on the phase space so that:

$$\langle\, f\,\rangle_{\mathcal{N}} = \int f(x,t)\,\mathcal{N}(x,t)\,dx = f\left(\{x_i(t)\}, t\right) \tag{12}$$

Taking into account the definition (10) it follows that:

$$\frac{\partial\mathcal{N}}{\partial t} + \frac{\partial(\mathbf{v}\mathcal{N})}{\partial\mathbf{r}} + \frac{\partial\left(\dot{\mathbf{v}}\mathcal{N}\right)}{\partial\mathbf{v}} = 0\,. \tag{13}$$

This is exactly the conservation law due to Eq. (11). Note that because the interaction between particle is general is not potential and depends explicitly on their velocities Eq. (13) differs from that for molecular systems. In the case of the equation of motion (1) we get:

$$\frac{\partial\mathcal{N}(x,t)}{\partial t} + \mathbf{v}\frac{\partial\mathcal{N}(x,t)}{\partial\mathbf{r}} + \gamma\frac{\partial}{\partial\mathbf{v}}\left([[\mathbf{v}, \int dx'\mathcal{N}(x',t)\mathbf{v}'K(\mathbf{r}-\mathbf{r}')], \mathbf{v}], \mathcal{N}(x,t)\right) = 0 \tag{14}$$

From this equation one can come to the equation for one-particle distribution function using the averaging procedure, which uses that $\overline{\mathcal{N}(x,t)} = n f_1(x,t)$, where $n \equiv N/L^2$. Lets define the mean velocity of the neighbors $\mathbf{w}(\mathbf{r}, t)$ as:

$$\rho(\mathbf{r},t)\mathbf{w}(\mathbf{r},t) \equiv n\int K(\mathbf{r}-\mathbf{r}')\mathbf{v}' f_1(x',t)dx'$$

Then the force field is

$$\mathbf{F}(x,t) = \gamma\rho(\mathbf{r},t)[[\mathbf{v}, \mathbf{w}(\mathbf{r},t)], \mathbf{v}] \tag{15}$$

In the approach when one neglects the correlation we get:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial\mathbf{r}} + \gamma\rho(\mathbf{r},t)\frac{\partial}{\partial\mathbf{v}}[[\mathbf{v}, \mathbf{w}(\mathbf{r},t)], \mathbf{v}]\right) f_1(x,t) = 0 \tag{16}$$

Neglecting the spatial variation of the local velocity of the neighbors $\mathbf{w}$ the standard form of the kinetic equation for the one-particle distribution function f_1 is obtained:

$$\frac{\partial f_1(x,t)}{\partial t} + \gamma n w\frac{\partial}{\partial\theta}(\sin(\phi - \theta) f_1(x,t)) = 0 \tag{17}$$

which is equivalent to the Langevin equation (5) without noise.

We have also performed an experiment with a system of self-propelled particles that were placed in boundary conditions corresponding to the standard Couette problem. Left and right border were periodic, as it is used usually in the simulations. Bottom border was just reflecting, and top border was making particles go in one defined direction (to the right). So, when the particle came close enough to the border it was turned to follow the specified direction. Then we made an experiment as Couette's flow, to study the velocity profile and the viscosity properties of this “liquid”. The system has the following parameters: Number of particles $N = 4000$, size of the system $L = 30$ strength of interaction $gamma = 1$, total number of time steps of a simulation $t = 10000$, velocity $v = 1$, size of time step $dt = 0.1$. The resulting velocity profiles for different noise strength values η are shown on the Fig. 2. The results show trivial profile of the velocity. This means that the hydrodynamics is dissipative-free.

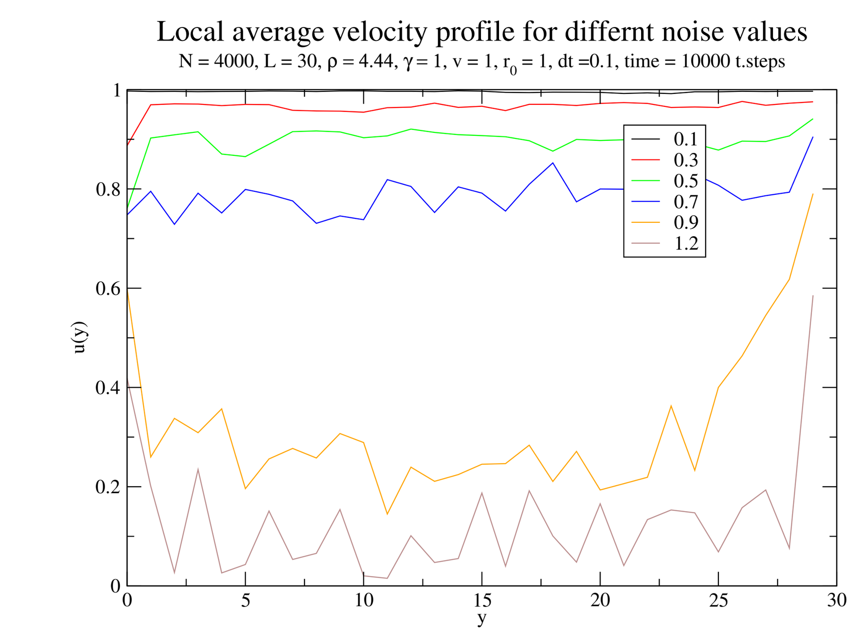

Figure 2: *The velocity profile of the “Coutte flow“ geometry determined for different (scalar) noise intensities*